

# BLOCK-GÖTTSCHE INVARIANTS FROM WALL-CROSSING

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**ABSTRACT.** We show how some of the refined tropical counts of Block and Göttsche emerge from the wall-crossing formalism. This leads naturally to a definition of a class of putative  $q$ -deformed Gromov-Witten invariants. We prove that this coincides with another natural  $q$ -deformation, provided by a result of Reineke and Weist in the context of quiver representations, when the latter is well defined.

## 1. INTRODUCTION

Recently Block and Göttsche [BG] (see also the accounts in [SG] section 6 and [IM] section 1) introduced a refined tropical count for plane tropical curves, where the usual Mikhalkin multiplicity is replaced by a function taking values in Laurent polynomials in one variable. The original motivation for Block and Göttsche's proposal is connected with a generalization of Göttsche's conjecture and a refinement of Severi degrees. The tropical invariance of such counts was proved by Itenberg and Mikhalkin [IM]. This invariance is perhaps surprising from a purely tropical point of view.

The purpose of this note is to point out a different perspective from which the definition of the Block-Göttsche multiplicity and the invariance of some of the associated refined tropical counts look completely natural. This point of view is provided by the wall-crossing formula for refined Donaldson-Thomas invariants.

In section 2 we recall a method due to Gross, Pandharipande and Siebert [GPS] (based on the tropical vertex group  $\mathbb{V}$ ) that allows to express the wall-crossing of numerical Donaldson-Thomas invariants<sup>1</sup> in terms of invariants enumerating *rational* tropical curves; the fundamental example is given by formulae (2.2) and (2.4) below. At the end of section 2 we motivate the need to go from numerical to refined wall-crossing formulae: this is natural from the Donaldson-Thomas point of view (the main reference being the work of Kontsevich and Soibelman [KS]). The details of the  $q$ -deformation are given in section 3. In section 4 we show that the definition of the Block-Göttsche multiplicity, and the invariance of the related counts for rational tropical curves, are essentially equivalent to an extension of the methods of [GPS] sections 1 and 2 to the refined wall-crossing formulae. This is summarized in Corollary 4.8. In section 5 (Proposition 5.1) we finally give the refinement of the basic GPS formula (2.4).

An important point is that this approach leads naturally to Gromov-Witten theory. Indeed [GPS] shows that computing commutators in  $\mathbb{V}$  is *equivalent* to the calculation of a class of genus zero Gromov-Witten invariants  $N[(\mathbf{P}_1, \mathbf{P}_2)]$  of (blowups of) weighted projective planes, parametrized by a pair of partitions  $(\mathbf{P}_1, \mathbf{P}_2)$  (see (2.3) and (2.6) below). By adapting this argument to the  $q$ -deformed case we find a natural  $q$ -deformation of  $N[(\mathbf{P}_1, \mathbf{P}_2)]$  in terms of Block-Göttsche counts, namely Definition 5.2. For some special values of  $(\mathbf{P}_1, \mathbf{P}_2)$  a result of Reineke and Weist [RW] shows that

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<sup>1</sup>This method is very general and also works for factorizations in  $\mathbb{V}$  which do not correspond to some wall-crossing.

$N[(\mathbf{P}_1, \mathbf{P}_2)]$  equals the Euler characteristic of a suitable moduli space of quiver representations  $\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2)$ . A different choice of  $q$ -deformation is then the symmetrized Poincaré polynomial  $q^{-\frac{1}{2} \dim \mathcal{M}(\mathbf{P}_1, \mathbf{P}_2)} P(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))(q)$ . We show that the two choices coincide (Theorem 5.3). A key ingredient is the Manschot-Pioline-Sen formula [MPS]; indeed it follows from the proof of Theorem 5.3 that the MPS formula in this context can be interpreted precisely as the equality of the two “quantizations”.

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## 2. TROPICAL VERTEX

The tropical vertex group  $\mathbb{V}$  is a subgroup of the group of formal 1-parameter families of automorphisms of the complex algebraic torus  $\mathbb{C}^* \times \mathbb{C}^*$ ,  $\mathbb{V} \subset \text{Aut}_{\mathbb{C}[[t]]} \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$ . Fix integers  $a, b$  and a function  $f_{(a,b)} \in \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$  of the form

$$f_{(a,b)} = 1 + t x^a y^b g(x^a y^b, t) \quad (2.1)$$

for a formal power series  $g \in \mathbb{C}[z][[t]]$ . To this we attach an element  $\theta_{(a,b), f_{(a,b)}} \in \mathbb{V}$  defined by

$$\theta_{(a,b), f_{(a,b)}}(x) = x f_{(a,b)}^{-b}, \quad \theta_{(a,b), f_{(a,b)}}(y) = y f_{(a,b)}^a.$$

Then we define  $\mathbb{V}$  as the completion with respect to  $(t) \subset \mathbb{C}[[t]]$  of the subgroup of  $\text{Aut}_{\mathbb{C}[[t]]} \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$  generated by all the transformations  $\theta_{(a,b), f_{(a,b)}}$  (as  $(a, b)$  and  $f_{(a,b)}$  vary). Elements of  $\mathbb{V}$  are “formal symplectomorphisms”, i.e. they preserve the holomorphic symplectic form  $\frac{dx}{x} \wedge \frac{dy}{y}$ . The basic question about  $\mathbb{V}$  is to compute a general commutator,  $[\theta_{(a_1, b_1), f_1}, \theta_{(a_2, b_2), f_2}] = \theta_{(a_2, b_2), f_2}^{-1} \theta_{(a_1, b_1), f_1} \theta_{(a_2, b_2), f_2} \theta_{(a_1, b_1), f_1}^{-1}$ . Despite its elementary flavour, it turns out that this problem plays a crucial role in a number of contexts in algebraic geometry, most importantly for us in wall-crossing formulae for counting invariants in abelian and triangulated categories (see [KS]). Suppose for definiteness that  $a_1, b_1, a_2, b_2$  are all nonnegative, and that  $(a_1, b_1)$  follows  $(a_2, b_2)$  in clockwise order. Then there exists a unique, possibly infinite (but countable) collection of *primitive* vectors  $(a, b)$  with positive entries, and attached functions  $f_{(a,b)}$  (of the form (2.1)) such that  $\theta_{(a_2, b_2), f_2}^{-1} \theta_{(a_1, b_1), f_1} \theta_{(a_2, b_2), f_2} \theta_{(a_1, b_1), f_1}^{-1} = \prod_{(a,b)}^{\rightarrow} \theta_{(a,b), f_{(a,b)}}$ . Here  $\prod^{\rightarrow}$  denotes the operation of writing products of finite subcollections of group elements  $\theta_{(a,b), f_{(a,b)}}$  from left to right so that the slopes of  $(a, b)$  in  $\mathbb{R}^2$  are decreasing (i.e. in clockwise order), and then taking the direct limit over all finite collections. Gross, Pandharipande and Siebert have shown that the problem of computing the functions  $f_{(a,b)}$  carries a surprisingly rich intrinsic geometry, which involves the virtual counts of rational curves in weighted projective planes with prescribed singularities and tangencies. To formulate the simplest result of this type, we fix two integers  $\ell_1, \ell_2$  and consider the transformations  $\theta_{(1,0), (1+tx)^{\ell_1}}, \theta_{(0,1), (1+ty)^{\ell_2}}$ . Let us define functions  $f_{(a,b)}$  as above (in particular, for  $(a, b)$  primitive) by

$$[\theta_{(1,0), (1+tx)^{\ell_1}}, \theta_{(0,1), (1+ty)^{\ell_2}}] = \prod_{(a,b)}^{\rightarrow} \theta_{(a,b), f_{(a,b)}}. \quad (2.2)$$

Since  $f_{(a,b)}$  has the form (2.1) we may take its logarithm as a formal power series, which must then be of the form  $\log f_{(a,b)} = \sum_{k \geq 0} c_k^{(a,b)} (tx)^{ak} (ty)^{bk}$ . Let us write  $\mathbf{P}$  for an

ordered partition and  $|\mathbf{P}|$  for its size (the sum of all its parts). Theorem 0.1 of [GPS] gives a formula for the coefficients  $c_k^{(a,b)}$  in terms of certain Gromov-Witten invariants  $N_{(a,b)}[(\mathbf{P}_a, \mathbf{P}_b)] \in \mathbb{Q}$ ,

$$c_k^{(a,b)} = k \sum_{|\mathbf{P}_a|=ka} \sum_{|\mathbf{P}_b|=kb} N_{(a,b)}[(\mathbf{P}_a, \mathbf{P}_b)], \quad (2.3)$$

where the length of  $\mathbf{P}_a$  ( $\mathbf{P}_b$ ) is  $\ell_1$  (respectively  $\ell_2$ ). Here  $N_{(a,b)}[(\mathbf{P}_a, \mathbf{P}_b)]$  is the virtual count of rational curves contained in the weighted projective plane  $\mathbb{P}(a, b, 1)$ , which must have prescribed singular points along the two toric divisors  $D_1, D_2$  dual to the rays spanned by  $(-1, 0)$  and  $(0, -1)$  respectively, lying away from the torus fixed points, and with multiplicities specified by the ordered partitions  $\mathbf{P}_a, \mathbf{P}_b$ . To make this rigorous one blows up a number of fixed points on  $D_1, D_2$  and imposes a suitable degree condition. Moreover one has to make sense of Gromov-Witten theory away from the torus fixed points. See [GPS] Section 0.4 for a precise definition. Additionally the curves must be tangent to order  $k$  (at an unspecified point) to the divisor  $D_{\text{out}}$  dual to the ray spanned by  $(a, b)$ .

The equality (2.3) actually arises from the enumeration of certain plane tropical curves. Consider a *weight vector*  $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ , where each  $\mathbf{w}_i$  is a collection of integers  $w_{ij}$  (for  $1 \leq i \leq 2$  and  $1 \leq j \leq l_i$ ) such that  $1 \leq w_{i1} \leq w_{i2} \leq \dots \leq w_{il_i}$ . For  $1 \leq j \leq l_1$  choose a general collection of parallel lines  $\mathfrak{d}_{1j}$  in the direction  $(1, 0)$ , respectively  $\mathfrak{d}_{2j}$  in the direction  $(0, 1)$  for  $1 \leq j \leq l_2$ . We attach the weight  $w_{ij}$  to the line  $\mathfrak{d}_{ij}$ , and think of the lines  $\mathfrak{d}_{ij}$  as “incoming” unbounded edges for connected, rational tropical curves  $\Upsilon \subset \mathbb{R}^2$ . We prescribe that such curves  $\Upsilon$  have a single additional “outgoing” unbounded edge in the direction  $(|\mathbf{w}_1|, |\mathbf{w}_2|)$ . Let us denote by  $\mathcal{S}(\mathbf{w})$  the finite set of such tropical curves  $\Upsilon$  (for a general, fixed choice of ends  $\mathfrak{d}_{ij}$ ). Let  $(|\mathbf{w}_1|, |\mathbf{w}_2|) = (ka, kb)$  for some positive integer  $k$  and primitive  $(a, b)$ . We denote by  $N_{(a,b)}^{\text{trop}}(\mathbf{w}) = \#^\mu \mathcal{S}(\mathbf{w})$  the tropical count of curves  $\Upsilon$  as above, i.e. the number of elements of  $\mathcal{S}(\mathbf{w})$  counted with the usual multiplicity  $\mu$  of tropical geometry (see [M]). It is known that  $\#^\mu \mathcal{S}(\mathbf{w})$  does not depend on the general choice of unbounded edges  $\mathfrak{d}_{ij}$  (see [M], [GM]). An application of [GPS] Theorem 2.8 gives

$$c_k^{(a,b)} = k \sum_{|\mathbf{P}_a|=ka} \sum_{|\mathbf{P}_b|=kb} \sum_{\mathbf{w}} \prod_{i=1}^2 \frac{R_{\mathbf{P}_i|\mathbf{w}_i}}{|\text{Aut}(\mathbf{w}_i)|} N_{(a,b)}^{\text{trop}}(\mathbf{w}), \quad (2.4)$$

where the inner sum is over weight vectors  $\mathbf{w}$  such that  $|\mathbf{w}_i| = \mathbf{P}_i$  and  $R_{\mathbf{P}_i|\mathbf{w}_i}, |\text{Aut}(\mathbf{w}_i)|$  are certain combinatorial coefficients. The connection to Gromov-Witten theory is established through the identity

$$N_{(a,b)}[(\mathbf{P}_a, \mathbf{P}_b)] = \sum_{\mathbf{w}} \prod_{i=1}^2 \frac{R_{\mathbf{P}_i|\mathbf{w}_i}}{|\text{Aut}(\mathbf{w}_i)|} N_{(a,b)}^{\text{trop}}(\mathbf{w}). \quad (2.5)$$

which follows from [GPS] Theorems 3.4, 4.4 and Proposition 5.3. In fact the invariants  $N_{(a,b)}[(\mathbf{P}_a, \mathbf{P}_b)]$  are completely determined by factorizations in an extended tropical vertex group. Introducing auxiliary variables  $s_1, \dots, s_{\ell_1}, t_1, \dots, t_{\ell_2}$  one considers the problem of computing the commutator  $[\prod_{i=1}^{\ell_1} \theta_{(1,0),1+s_i x}, \prod_{j=1}^{\ell_2} \theta_{(0,1),1+t_j y}]$ , with the obvious extension of the notation introduced above. Then one can refine (2.3) to show that the the corresponding weight functions satisfy

$$\log f_{(a,b)} = k \sum_{|\mathbf{P}_a|=ka} \sum_{|\mathbf{P}_b|=kb} N_{(a,b)}[(\mathbf{P}_a, \mathbf{P}_b)] s^{\mathbf{P}_a} t^{\mathbf{P}_b} x^{ka} y^{kb}. \quad (2.6)$$

Now we make the basic observation that operators of the form of  $\theta_1 = \theta_{(1,0),(1+tx)^{\ell_1}}$  and  $\theta_2 = \theta_{(0,1),(1+ty)^{\ell_2}}$  admit natural  $q$ -deformations or “quantizations”, acting on the  $\mathbb{C}[[t]]$ -algebra generated by quantum variables  $\hat{x}\hat{y} = q\hat{y}\hat{x}$ . This is motivated by their special role in Donaldson-Thomas theory, where they represents the action of a stable spherical object ([KS] section 6.4). Roughly speaking then  $q$ -deformation corresponds to passing from Euler characteristics of moduli spaces to their Poincaré polynomials. More generally in mathematical physics these operators reflect the spectrum of BPS states of theories belonging to a suitable class, and their  $q$ -deformation is connected with refined indices counting such states (see e.g. [P] for an introduction to this circle of ideas). In the present simple example  $\hat{\theta}_1(\hat{x}) = \hat{x}$ ,  $\hat{\theta}_1(\hat{y}) = \hat{y}(1 + q^{\frac{1}{2}}t\hat{x})^{\ell_1}$ ,  $\hat{\theta}_2(\hat{x}) = \hat{x}(1 + q^{\frac{1}{2}}t\hat{y})^{-\ell_2}$ ,  $\hat{\theta}_2(\hat{y}) = \hat{y}$ . It is then natural to guess the existence of a  $q$ -deformation of the factorization (2.2) for  $[\hat{\theta}_1, \hat{\theta}_2]$ , as well as of a  $q$ -analogue of (2.4). From the form of (2.4) one may envisage the existence of  $q$ -deformed tropical counts  $\hat{N}_{(a,b)}^{\text{trop}}(\mathbf{w})$ , which should be defined as  $\#^{\mu_q} \mathcal{S}(\mathbf{w})$  for some  $q$ -deformation of the usual tropical multiplicity. We will see in section 4 that this is precisely what happens:  $\mu_q$  turns out to be the Block-Göttsche multiplicity. Another advantage of this point of view is that (2.5) immediately suggests the form of some putative  $q$ -deformed Gromov-Witten invariants. We will discuss this in section 5.

### 3. $q$ -DEFORMATION

We will need a more general incarnation of the group  $\mathbb{V}$ . Let  $R$  be a  $\mathbb{C}$ -algebra which is either complete local or Artin. Let  $\Gamma$  be a fixed lattice with an antisymmetric, bilinear form  $\langle -, - \rangle$ . Consider the infinite dimensional complex Lie algebra  $\mathfrak{g}$  generated by  $e_\alpha, \alpha \in \Gamma$ , with bracket

$$[e_\alpha, e_\beta] = \langle \alpha, \beta \rangle e_{\alpha+\beta}. \quad (3.1)$$

We also endow  $\mathfrak{g}$  with the associative, commutative product determined by

$$e_\alpha e_\beta = e_{\alpha+\beta}. \quad (3.2)$$

With the product (3.2) and the bracket (3.1)  $\mathfrak{g}$  becomes a Poisson algebra: the linear map  $[x, -]$  satisfies the Leibniz rule (this is a straightforward check on the generators). We write  $\hat{\mathfrak{g}}$  for the completed tensor product of  $\mathfrak{g}$  with  $R$ ,  $\hat{\mathfrak{g}} = \mathfrak{g} \hat{\otimes}_{\mathbb{C}} R = \varprojlim \mathfrak{g} \otimes_{\mathbb{C}} R/\mathfrak{m}_R^k$ , and extend the Poisson structure to  $\hat{\mathfrak{g}}$  by  $R$ -linearity. Let  $f_\alpha \in \hat{\mathfrak{g}}$  be an element of the form

$$f_\alpha \in 1 + \mathfrak{m}_R[e_\alpha]e_\alpha. \quad (3.3)$$

Let us introduce a class of Poisson automorphisms  $\theta_{\alpha, f_\alpha}$  of the  $R$ -algebra  $\hat{\mathfrak{g}}$  by prescribing

$$\theta_{\alpha, f_\alpha}(e_\beta) = e_\beta f_\alpha^{\langle \alpha, \beta \rangle}. \quad (3.4)$$

Notice that the inverse automorphism  $\theta_{\alpha, f_\alpha}^{-1}$  is given by  $\theta_{\alpha, f_\alpha^{-1}}$ . More generally for  $\Omega \in \mathbb{Q}$  we denote by  $\theta_{\alpha, f_\alpha}^\Omega$  the automorphism  $\theta_{\alpha, f_\alpha^\Omega}$ .

**Definition 3.1.** *The tropical vertex group  $\mathbb{V}_{\Gamma, R}$  is the completion with respect to  $\mathfrak{m}_R \subset R$  of the subgroup of  $\text{Aut}_R(\hat{\mathfrak{g}})$  generated by all the transformations  $\theta_{\alpha, f_\alpha}$  (as  $\alpha$  varies in  $\Gamma$  and  $f_\alpha$  among functions of the form (3.3)).*

**Lemma 3.2.** *Suppose that  $\Gamma$  is the lattice  $\mathbb{Z}^2$  endowed with its standard antisymmetric form  $\langle (p, q), (p', q') \rangle = pq' - qp'$ . Then  $\mathbb{V}_{\Gamma, \mathbb{C}[[t]]} \cong \mathbb{V}$ .*

*Proof.* We can identify  $\mathbb{C}[x, x^{-1}, y, y^{-1}]$  with  $\mathfrak{g}$  by the isomorphism  $\iota$  defined by  $\iota(x) = e_{(1,0)}$ ,  $\iota(y) = e_{(0,1)}$ . Taking  $\hat{\otimes}_{\mathbb{C}[[t]]}$  on both sides,  $f_{(a,b)} \in 1 + x^a y^b \mathfrak{m}_{\mathbb{C}[[t]]}[x^a y^b]$  is mapped

to some  $f_\alpha \in 1 + \mathfrak{m}_{\mathbb{C}[[t]]}[e_\alpha]e_\alpha$  where  $\alpha = (a, b)$ , and by (3.4) we have  $\iota^{-1} \circ \theta_{\alpha, f_\alpha} \circ \iota = \theta_{(a, b), f_{a, b}}$ . This proves the claim since  $\theta_{(a, b), f_{a, b}}$ ,  $\theta_{\alpha, f_\alpha}$  are topological generators.  $\square$

Elements of  $\mathbb{V}_{\Gamma, R}$  of the form  $\theta_{\alpha, 1 + \sigma e_\alpha}$  with  $\sigma \in \mathfrak{m}_R$  play a special role, as they have a natural interpretation in Donaldson-Thomas theory and in the mathematical physics of BPS states. In particular as we will see they have a well-defined  $q$ -deformation. Accordingly we can give a definition of the subgroup of  $\mathbb{V}_{\Gamma, R}$  which is relevant to the wall-crossing of Donaldson-Thomas invariants.

**Definition 3.3.** *The wall-crossing group  $\tilde{\mathbb{V}}_{\Gamma, R} \subset \mathbb{V}_{\Gamma, R}$  is the completion of the subgroup generated by automorphisms of the form  $\theta_{\alpha, 1 + \sigma e_\alpha}^\Omega$  for  $\alpha \in \Gamma$ ,  $\sigma \in \mathfrak{m}_R$  and  $\Omega \in \mathbb{Q}$ .*

An argument in [GPS] Section 1 implies that we do not lose too much by restricting to groups of the form  $\tilde{\mathbb{V}}_{\Gamma, S}$ , provided we work over a suitably large ring  $S$ . The argument bears on the case when  $R = \mathbb{C}[[t_1, \dots, t_n]]$  and  $f_\alpha$  is of the form  $f_\alpha = 1 + t_i e_\alpha g(e_\alpha, t_i)$  for some  $i = 1, \dots, n$  and  $g \in \mathbb{C}[z][[t_i]]$ . First we work modulo  $\mathfrak{m}_R^{k+1}$  for some  $k$ : this will be the order of approximation. Then over  $R_k = R/(t_1^{k+1}, \dots, t_n^{k+1})$  we have

$$\log f_\alpha = \sum_{j=1}^k \sum_{w \geq 1} w a_{ijw} e_{w\alpha} t_i^k \quad (3.5)$$

for some coefficients  $a_{ijw}$  which vanish for all but finitely many  $w$ . We introduce variables  $u_{ij}$  for  $i = 1, \dots, n$ ,  $j = 1, \dots, k$ , and pass to the base ring  $\tilde{R}_k = \mathbb{C}[u_{ij}]/(u_{ij}^2)$ . There is an inclusion  $i: R_k \hookrightarrow \tilde{R}_k$  given by  $t_i \mapsto \sum_{j=1}^k u_{ij}$ . In particular we have an inclusion of groups  $\mathbb{V}_{\Gamma, R_k} \hookrightarrow \mathbb{V}_{\Gamma, \tilde{R}_k}$ . We will often be sloppy and identify an element of  $R_k$  with its image under  $i$ . As  $u_{ij}^2 = 0$ , the image of (3.5) under  $i$  is

$$\log f_\alpha = \sum_{j=1}^k \sum_{\#J=j} \sum_{w \geq 1} j! w a_{ijw} \prod_{l \in J} u_{il} e_{w\alpha},$$

summing over  $J \subset \{1, \dots, n\}$ . Exponentiating both sides and using again  $u_{ij}^2 = 0$  we find the factorization

$$f_\alpha = \prod_{j=1}^k \prod_{\#J=j} \prod_{w \geq 1} \left( 1 + j! w a_{ijw} \prod_{l \in J} u_{il} e_{w\alpha} \right).$$

So for  $f_\alpha$  as above we have found a factorization in  $\mathbb{V}_{\Gamma, \tilde{R}_k}$

$$\theta_{\alpha, f_\alpha} \equiv \prod_{j=1}^k \prod_{\#J=j} \prod_{w \geq 1} \theta_{\alpha, f_{jJw}} \bmod t_i^{k+1}, \quad f_{jJw} = 1 + j! w a_{ijw} \prod_{l \in J} u_{il} e_{w\alpha}. \quad (3.6)$$

Notice that we have  $\theta_{\alpha, f_{jJw}} = \theta_{w\alpha, 1 + j! a_{ijw} \prod_{l \in J} u_{il} e_{w\alpha}}$ .

Following [KS], we can take advantage of the Poisson structure on  $\hat{\mathfrak{g}}$  to give a different expression for the special transformations  $\theta_{\alpha, 1 + \sigma e_{m\alpha}}$ , which leads easily to their  $q$ -deformation. Fix  $\sigma \in \mathfrak{m}_R$ , and define the *dilogarithm*  $\text{Li}_2(\sigma e_\alpha)$  by  $\text{Li}_2(\sigma e_\alpha) = \sum_{k \geq 1} \frac{\sigma^k e_{k\alpha}}{k^2}$ . This is well defined by our assumptions on  $R$ . Then  $\text{ad}(\text{Li}_2(\sigma e_\alpha)) = [\text{Li}_2(\sigma e_\alpha), -]$  is a derivation of  $\hat{\mathfrak{g}}$ , and again by our assumptions on  $R$  its exponential is a well defined Poisson automorphism of  $\hat{\mathfrak{g}}$ , acting by

$$\exp(\text{ad}(\text{Li}_2(\sigma e_\alpha)))(e_\beta) = \sum_{h \geq 0} \frac{1}{h!} \text{ad}^h(\text{Li}_2(\sigma e_\alpha))(e_\beta).$$

**Lemma 3.4.** *The automorphism  $\theta_{\alpha, 1 + \sigma e_{m\alpha}}$  is the same as  $\exp(\frac{1}{m} \text{ad}(\text{Li}_2(-\sigma e_{m\alpha})))$ .*

*Proof.* We have

$$\begin{aligned} \left[\frac{1}{m} \text{Li}_2(-\sigma e_{m\alpha}), e_\beta\right] &= \frac{1}{m} \sum_{k \geq 1} (-1)^k \frac{\sigma^k}{k^2} [e_{km\alpha}, e_\beta] = \frac{1}{m} \sum_{k \geq 1} (-1)^k \frac{\sigma^k}{k^2} \langle km\alpha, \beta \rangle e_{km\alpha+\beta} \\ &= \sum_{k \geq 1} (-1)^k \frac{\sigma^k}{k} \langle \alpha, \beta \rangle e_\beta e_{km\alpha} = e_\beta \langle \alpha, \beta \rangle \sum_{k \geq 1} (-1)^k \frac{\sigma^k e_{km\alpha}}{k} \\ &= e_\beta \langle \alpha, \beta \rangle \log(1 + \sigma e_{m\alpha}). \end{aligned}$$

Using the Leibniz rule and induction, we find

$$\text{ad}^h\left(\frac{1}{m} \text{Li}_2(-\sigma e_{m\alpha})\right)(e_\beta) = e_\beta (\langle \alpha, \beta \rangle \log(1 + \sigma e_{m\alpha}))^h,$$

and the result follows.  $\square$

We now replace  $\mathfrak{g}$  with the associative, noncommutative algebra  $\mathfrak{g}_q$  over  $\mathbb{C}(q^{\pm \frac{1}{2}})$ , generated by  $\hat{e}_\alpha, \alpha \in \Gamma$ . The classical product (3.2) is quantized to

$$\hat{e}_\alpha \hat{e}_\beta = q^{\frac{1}{2}\langle \alpha, \beta \rangle} \hat{e}_{\alpha+\beta}. \quad (3.7)$$

The classical limit<sup>2</sup> is obtained when we let  $q^{\frac{1}{2}} \rightarrow 1$ . As standard in the quantization the Lie bracket is the natural one given by the commutator. In other words we are now thinking of the  $\hat{e}_\alpha$  as operators (as opposed to the classical bracket (3.1), which corresponds to a Poisson bracket of the  $e_\alpha$  seen as functions). Namely, we set

$$[\hat{e}_\alpha, \hat{e}_\beta] := (q^{\frac{1}{2}\langle \alpha, \beta \rangle} - q^{-\frac{1}{2}\langle \alpha, \beta \rangle}) \hat{e}_{\alpha+\beta}. \quad (3.8)$$

Since this is the commutator bracket of an associative algebra,  $\mathfrak{g}_q$  is automatically Poisson. After rescaling, the Lie bracket (3.8) has the classical limit (3.1):

$$\lim_{q^{\frac{1}{2}} \rightarrow 1} \frac{1}{q-1} [\hat{e}_\alpha, \hat{e}_\beta] = \langle \alpha, \beta \rangle \hat{e}_{\alpha+\beta}.$$

Fixing a local complete or Artin  $\mathbb{C}$ -algebra  $R$  as usual, we define  $\hat{\mathfrak{g}}_q = \mathfrak{g}_q \hat{\otimes}_{\mathbb{C}} R$ . The fundamental case is  $\mathfrak{g}_q[[t]]$  where  $t$  is a central variable.

According to Lemma 3.4 the action of  $\theta_{\alpha, 1+\sigma e_\alpha}$  on  $\hat{\mathfrak{g}}$  is the same as the adjoint action of  $\exp(\text{ad}(\text{Li}_2(-\sigma e_\alpha)))$ . So we need to find an element of  $\hat{\mathfrak{g}}_q$  which plays the role of the (exponential of the) dilogarithm. This is the  $q$ -dilogarithm,

$$\mathbf{E}(\sigma \hat{e}_\alpha) = \sum_{n \geq 0} \frac{(-q^{\frac{1}{2}} \sigma \hat{e}_\alpha)^n}{(1-q)(1-q^2) \cdots (1-q^n)}.$$

This only involves commuting variables, and is well defined by our assumptions on  $R$ . The  $q$ -dilogarithm is in fact a  $q$ -deformation of  $\exp(\text{ad}(\text{Li}_2(\sigma e_\alpha)))$ , as shown by the standard rewriting

$$\mathbf{E}(\sigma \hat{e}_\alpha) = \exp \left( - \sum_{k \geq 1} \frac{\sigma^k \hat{e}_{k\alpha}}{k((-q^{\frac{1}{2}})^k - (-q^{\frac{1}{2}})^{-k})} \right).$$

The  $q$ -dilogarithm also admits a well known infinite product expansion,

$$\mathbf{E}(\sigma \hat{e}_\alpha) = \prod_{k \geq 0} \left( 1 + q^{k+\frac{1}{2}} \sigma \hat{e}_\alpha \right)^{-1}. \quad (3.9)$$

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<sup>2</sup>This is the opposite of the classical limit considered in [KS]. It gives the wrong sign for BPS states counts, but it is the natural choice for the purposes of this note.

For  $\Omega \in \mathbb{Q}$  we introduce automorphisms  $\hat{\theta}^\Omega[\sigma\hat{e}_\alpha]$  of  $\hat{\mathfrak{g}}_q$  acting by

$$\hat{\theta}^\Omega[\sigma\hat{e}_\alpha](\hat{e}_\beta) = \text{Ad } \mathbf{E}^\Omega(\sigma\hat{e}_\alpha)(\hat{e}_\beta) = \mathbf{E}^\Omega(\sigma\hat{e}_\alpha)\hat{e}_\beta\mathbf{E}^{-\Omega}(\sigma\hat{e}_\alpha).$$

We regard  $\hat{\theta}[\sigma\hat{e}_\alpha]$  as the required quantization of  $\theta_{\alpha,1+\sigma e_\alpha}$ . Notice the change of notation from the classical case. This is more practical especially since we will also need to consider the shifted operators  $\hat{\theta}[\sigma(-q^{\frac{1}{2}})^n\hat{e}_\alpha] = \text{Ad } \mathbf{E}(\sigma(-q^{\frac{1}{2}})^n\hat{e}_\alpha)$  for  $n \in \mathbb{Z}$ . We can now prove the analogue of Lemma 3.4.

**Lemma 3.5.** *The adjoint action is given by*

$$\hat{\theta}^\Omega[\sigma(-q^{\frac{1}{2}})^n\hat{e}_\alpha](\hat{e}_\beta) = \hat{e}_\beta \prod_{k=0}^{\langle\alpha,\beta\rangle-1} \left(1 + (-1)^n q^{k+\frac{n+1}{2}} \sigma\hat{e}_\alpha\right)^\Omega \prod_{k=\langle\alpha,\beta\rangle}^{-1} \left(1 + (-1)^n q^{k+\frac{n+1}{2}} \sigma\hat{e}_\alpha\right)^{-\Omega}. \quad (3.10)$$

It is straightforward to check that (3.10) has the expected classical limit (3.4) as  $q^{\frac{1}{2}} \rightarrow 1$ .

*Proof.* Suppose to start with that  $n = 0$  and set  $\kappa = \langle\alpha, \beta\rangle$ . Then  $\hat{e}_\alpha\hat{e}_\beta = q^\kappa\hat{e}_\beta\hat{e}_\alpha$ , and so if  $f(\hat{e}_\alpha)$  is a formal power series with coefficients in  $\mathbb{C}(q^{\pm\frac{1}{2}})$ , we have

$$f(\hat{e}_\alpha)\hat{e}_\beta = \sum_{i \geq 0} a_i \hat{e}_{i\alpha} \hat{e}_\beta = \hat{e}_\beta \sum_{i \geq 0} a_i q^{i\kappa} \hat{e}_{i\alpha} = \hat{e}_\beta f(q^\kappa \hat{e}_\alpha).$$

Apply this to  $f(\hat{e}_\alpha) = \mathbf{E}^\Omega(\sigma\hat{e}_\alpha)$  to get

$$\hat{\theta}^\Omega[\sigma\hat{e}_\alpha](\hat{e}_\beta) = \mathbf{E}^\Omega(\sigma\hat{e}_\alpha)\hat{e}_\beta\mathbf{E}^{-\Omega}(\sigma\hat{e}_\alpha) = \hat{e}_\beta\mathbf{E}^\Omega(q^\kappa\sigma\hat{e}_\alpha)\mathbf{E}^{-\Omega}(\sigma\hat{e}_\alpha).$$

Suppose for a moment that  $\kappa \geq 0$  and use the product expansion (3.9) to get

$$\begin{aligned} \mathbf{E}^\Omega(q^\kappa\sigma\hat{e}_\alpha) &= \prod_{i \geq 0} \left(1 + q^{i+\frac{1}{2}} q^\kappa \sigma\hat{e}_\alpha\right)^{-\Omega} = \prod_{i \geq \kappa} \left(1 + q^{i+\frac{1}{2}} \sigma\hat{e}_\alpha\right)^{-\Omega} \\ &= \prod_{i=0}^{\kappa-1} \left(1 + q^{i+\frac{1}{2}} \hat{e}_\alpha\right)^\Omega \mathbf{E}^\Omega(\sigma\hat{e}_\alpha), \end{aligned}$$

and (3.10) follows. For  $n \neq 0$  we use similarly

$$\begin{aligned} \mathbf{E}(q^\kappa(-q^{\frac{1}{2}})^n\hat{e}_\alpha) &= \prod_{i \geq 0} \left(1 + (-1)^n q^{i+\frac{n+1}{2}} q^\kappa \hat{e}_\alpha\right)^{-\Omega} = \prod_{i \geq \kappa} \left(1 + (-1)^n q^{i+\frac{n+1}{2}} \hat{e}_\alpha\right)^{-\Omega} \\ &= \prod_{i=0}^{\kappa-1} \left(1 + (-1)^n q^{i+\frac{n+1}{2}} \hat{e}_\alpha\right)^\Omega \mathbf{E}^\Omega((-q^{\frac{1}{2}})^n\hat{e}_\alpha). \end{aligned}$$

The result for  $\kappa < 0$  also follows since  $\hat{\theta}[\sigma\hat{e}_\alpha]$  is an algebra automorphism.  $\square$

We can now introduce the  $q$ -deformed analogue of the group  $\tilde{\mathbf{V}}_R$ .

**Definition 3.6.**  $\mathbb{U}_{\Gamma,R}$  is the completion of the subgroup of  $\text{Aut}_{\mathbb{C}(q^{\pm\frac{1}{2}}) \otimes_{\mathbb{C}} R} \hat{\mathfrak{g}}_q$  generated by automorphisms of the form  $\hat{\theta}^\Omega[(-q^{\frac{1}{2}})^n\sigma\hat{e}_\alpha]$  (where  $\alpha \in \Gamma$ ,  $\sigma \in \mathfrak{m}_R$ ,  $\Omega \in \mathbb{Q}$ ,  $n \in \mathbb{Z}$ ), with respect to the  $\mathfrak{m}_R$ -adic topology.

The factorization (2.2) has an analogue in the  $q$ -deformed case. Suppose that  $\alpha_1$  follows  $\alpha_2$  in clockwise order.

**Lemma 3.7.** *Fix positive integers  $\ell_1, \ell_2$ . There exist unique  $\Omega_n(k\alpha) \in \mathbb{Q}$  such that*

$$[\hat{\theta}^{\ell_1}[\sigma_1 \hat{e}_{\alpha_1}], \hat{\theta}^{\ell_2}[\sigma_2 \hat{e}_{\alpha_2}]] = \prod_{\gamma}^{\rightarrow} \prod_{k \geq 1} \prod_{n \in \mathbb{Z}} \hat{\theta}^{(-1)^n \Omega_n(k\gamma)} [(-q^{\frac{1}{2}})^n \sigma^{k\gamma} \hat{e}_{k\gamma}], \quad (3.11)$$

where  $\prod^{\rightarrow}$  is a slope ordered product over primitive, positive vectors  $\gamma = \gamma^1 \alpha_1 + \gamma^2 \alpha_2$ , we set  $\sigma^{k\gamma} = \sigma_1^{k\gamma_1} \sigma_2^{k\gamma_2}$  and, for each fixed  $k$ ,  $\Omega_n(k\gamma)$  vanishes for all but finitely many  $n$ .

*Proof.* Use the Baker-Campbell-Hausdorff formula and induction on  $\gamma^1 + \gamma^2$  (see e.g. [P] section 2.1).  $\square$

Our main problem then becomes to find  $\hat{\theta}_{\gamma} = \prod_{k \geq 1} \prod_{n \in \mathbb{Z}} \hat{\theta}^{(-1)^n \Omega_n(k\gamma)} [(-q^{\frac{1}{2}})^n \sigma^{k\gamma} \hat{e}_{k\gamma}]$ . To compare this with (2.4) write  $\hat{x} = \hat{e}_{1,0}$ ,  $\hat{y} = \hat{e}_{0,1}$  (so  $\hat{x}\hat{y} = q\hat{y}\hat{x}$ ) and introduce a ‘‘Poincaré’’ Laurent polynomial in  $q^{\frac{1}{2}}$ ,  $P(k\gamma)(q) = \sum_{n \in \mathbb{Z}} (-1)^n \Omega_n(k\gamma) (-q^{\frac{1}{2}})^n$ . Then the action of  $\prod_{k \geq 1} \prod_{n \in \mathbb{Z}} \hat{\theta}^{(-1)^n \Omega_n(k\gamma)} [(-q^{\frac{1}{2}})^n \sigma^{k\gamma} \hat{e}_{k\gamma}]$  can be written as  $\hat{x} \mapsto \hat{x}f, \hat{y} \mapsto \hat{y}g$ , where  $f, g$  are commutative power series given by

$$\begin{aligned} \log f &= \sum_{m \geq 1} q^{-\frac{m^2}{2} \gamma^1 \gamma^2} (\sigma_1 \hat{x})^{m\gamma^1} (\sigma_2 \hat{y})^{m\gamma^2} \left( - \sum_{k|m} \sum_{s=-k\gamma^2}^{-1} q^{\frac{m}{k}s} \right) \hat{c}_m^{\gamma}, \\ \log g &= \sum_{m \geq 1} q^{-\frac{m^2}{2} \gamma^1 \gamma^2} (\sigma_1 \hat{x})^{m\gamma^1} (\sigma_2 \hat{y})^{m\gamma^2} \left( \sum_{k|m} \sum_{s=0}^{k\gamma^1-1} q^{\frac{m}{k}s} \right) \hat{c}_m^{\gamma} \end{aligned}$$

for coefficients  $\hat{c}_m^{\gamma} = \sum_{k|m} \frac{(-q^{\frac{1}{2}})^{\frac{m}{k}}}{\frac{m}{k}} P(k\gamma) (q^{\frac{m}{k}}) \in \mathbb{Q}[q^{\pm \frac{1}{2}}]$ .

#### 4. SCATTERING DIAGRAMS

We adapt the results of [GPS] sections 1 and 2 to the  $q$ -deformed setup. In this section we take  $\Gamma$  to be  $\mathbb{Z}^2$  with its standard antisymmetric bilinear form (as in Lemma 3.2).

**Definition 4.1.** A ray or line for  $\mathbb{U}_{\Gamma,R}$  is a pair  $(\mathfrak{d}, \hat{\theta}_{\mathfrak{d}})$  where

- $\mathfrak{d} \subset \Gamma \otimes \mathbb{R} = \mathbb{R}^2$  is a subset which is either of the form  $\alpha'_0 + \mathbb{R}_{\geq 0} \alpha_0$  (a ray), or  $\alpha'_0 + \mathbb{R} \alpha_0$  (a line), with  $\alpha'_0 \in \mathbb{R}^2$ , and  $\alpha_0 \in \Gamma$  positive;
- $\hat{\theta}_{\mathfrak{d}} \in \mathbb{U}_{\Gamma,R}$  is a (possibly infinite) product of elements of the form  $\hat{\theta}^{\Omega} [(-q^{\frac{1}{2}})^n \sigma \hat{e}_{k\alpha_0}]$ .

If  $\mathfrak{d}$  is a ray we write  $\partial \mathfrak{d}$  for its initial point, and set  $\partial \mathfrak{d} = \emptyset$  if  $\mathfrak{d}$  is a line.

**Definition 4.2.** A scattering diagram for  $\mathbb{U}_{\Gamma,R}$  is a collection of rays and lines  $(\mathfrak{d}, \hat{\theta}_{\mathfrak{d}})$  such that for every  $k \geq 1$  we have  $\hat{\theta}_{\mathfrak{d}} \equiv \text{Id} \bmod \mathfrak{m}_R^k$  for all but finitely many  $\mathfrak{d}$ .

The singular set of a scattering diagram  $\mathfrak{D}$  is  $\text{Sing}(\mathfrak{D}) = \bigcup_{\mathfrak{d} \in \mathfrak{D}} \partial \mathfrak{d} \cup \bigcup_{\dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = 0} \mathfrak{d}_1 \cap \mathfrak{d}_2$ . Let  $\pi: [0, 1] \rightarrow \mathbb{R}^2$  be a smooth path. We say that  $\pi$  is admissible if  $\pi$  misses the singular set  $\text{Sing}(\mathfrak{D})$  and is transversal to every  $\mathfrak{d} \in \mathfrak{D}$ . We will define a notion of path ordered product  $\hat{\theta}_{\pi, \mathfrak{D}}$  along  $\pi$ . Let  $k \geq 1$ . Then  $\pi$  meets (transversely) only finitely many  $\mathfrak{d}$  with  $\hat{\theta}_{\mathfrak{d}} \not\equiv \text{Id} \bmod \mathfrak{m}_R^k$ . We denote this ordered collection by  $\mathfrak{d}_1, \dots, \mathfrak{d}_s$ , and define a partial ordered product  $\hat{\theta}_{\pi, \mathfrak{D}}^{(k)} = \hat{\theta}_{\mathfrak{d}_1}^{\varepsilon_1} \circ \dots \circ \hat{\theta}_{\mathfrak{d}_s}^{\varepsilon_s}$ . Here  $\varepsilon_i = 1$  if  $\{\pi', \alpha_0\}$  is a positive basis of  $\mathbb{R}^2$  (where  $\alpha_0$  is the direction of  $\mathfrak{d}_i$ ), and  $\varepsilon_i = -1$  otherwise. Notice that the only ambiguity in  $\hat{\theta}_{\pi, \mathfrak{D}}^{(k)}$  happens when  $\dim \mathfrak{d}_i \cap \mathfrak{d}_{i+1} = 1$ . But then  $\hat{\theta}_{\mathfrak{d}_i}$  and  $\hat{\theta}_{\mathfrak{d}_{i+1}}$  commute, so in fact  $\hat{\theta}_{\pi, \mathfrak{D}}^{(k)}$  is well defined. We then let  $\hat{\theta}_{\pi, \mathfrak{D}} = \lim_{\rightarrow} \hat{\theta}_{\pi, \mathfrak{D}}^{(k)}$ , a well defined element of  $\mathbb{U}_{\Gamma,R}$ . We say that a scattering diagram  $\mathfrak{D}$  is *saturated* if  $\hat{\theta}_{\pi, \mathfrak{D}} = \text{Id}$  for all admissible, closed paths  $\pi$ .



Two scattering diagrams  $\mathfrak{D}, \mathfrak{D}'$  are *equivalent* if  $\hat{\theta}_{\pi, \mathfrak{D}} = \hat{\theta}_{\pi, \mathfrak{D}'}$  whenever  $\pi$  is admissible for both. A simple induction argument (adapted e.g. from the proof of [GPS] Theorem 1.4) shows that a scattering diagram  $\mathfrak{D}$  admits a *saturation*: a saturated scattering diagram  $S(\mathfrak{D})$  which is obtained by adding to  $\mathfrak{D}$  a collection of rays. Moreover  $S(\mathfrak{D})$  is unique up to equivalence.

Suppose that the  $\alpha_i$  are positive and that  $\alpha_1$  follows  $\alpha_2$  in clockwise order. Then computing the operators  $\hat{\theta}_\gamma$  for  $[\hat{\theta}^{\ell_1}[\sigma_1 \hat{e}_{\alpha_1}], \hat{\theta}^{\ell_2}[\sigma_2 \hat{e}_{\alpha_2}]]$  is equivalent to determining  $S(\mathfrak{D})$  for the scattering diagram  $\mathfrak{D} = \{(\mathbb{R}\alpha_1, \hat{\theta}^{\ell_1}[\sigma_1 \hat{e}_{\alpha_1}]), (\mathbb{R}\alpha_2, \hat{\theta}^{\ell_2}[\sigma_2 \hat{e}_{\alpha_2}])\}$ . To see this choose  $\pi$  to be a closed loop with  $\pi(0) = (-1, -1)$  and winding once around the origin in clockwise direction. In general computing  $S(\mathfrak{D})$  can be very hard. However there is a special case when saturation is straightforward. For  $m \in \mathbb{Z}$  we set  $[m]_q = \frac{q^{\frac{m}{2}} - q^{-\frac{m}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ , the usual  $q$ -number.

**Lemma 4.3.** *Suppose  $\mathfrak{m}_R$  contains elements  $\sigma_1, \sigma_2$  with  $\sigma_i^2 = 0$ , and let*

$$\mathfrak{D} = \{(\mathbb{R}\alpha_1, \hat{\theta}[\sigma_1 \hat{e}_{\alpha_1}]), (\mathbb{R}\alpha_2, \hat{\theta}[\sigma_2 \hat{e}_{\alpha_2}])\}.$$

*Suppose that the  $\alpha_i$  are positive and that  $\alpha_1$  follows  $\alpha_2$  in clockwise order. Then  $S(\mathfrak{D})$  is obtained by adding the single ray*

$$(\mathbb{R}(\alpha_1 + \alpha_2), \hat{\theta}[\langle \alpha_1, \alpha_2 \rangle]_q \sigma_1 \sigma_2 \hat{e}_{\alpha_1 + \alpha_2}]).$$

*Proof.* Since  $\sigma_i^2 = 0$  we have

$$\hat{\theta}[\sigma_i \hat{e}_{\alpha_i}] = \text{Ad } \mathbf{E}(\sigma_i \hat{e}_{\alpha_i}) = \text{Ad } \exp\left(\frac{\sigma_i \hat{e}_{\alpha_i}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right).$$

Applying the Baker-Campbell-Hausdorff formula we find

$$\hat{\theta}[\sigma_1 \hat{e}_{\alpha_1}] \hat{\theta}[\sigma_2 \hat{e}_{\alpha_2}] = \text{Ad } \exp\left(\frac{\sigma_1 \hat{e}_{\alpha_1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} + \frac{\sigma_2 \hat{e}_{\alpha_2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} + \frac{1}{2}[\langle \alpha_1, \alpha_2 \rangle]_q \frac{\sigma_1 \sigma_2 \hat{e}_{\alpha_1 + \alpha_2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right).$$

Composing on the left with  $\hat{\theta}^{-1}[\sigma_2 \hat{e}_{\alpha_2}]$  gives  $\text{Ad } \exp\left(\frac{\sigma_1 \hat{e}_{\alpha_1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} + [\langle \alpha_1, \alpha_2 \rangle]_q \frac{\sigma_1 \sigma_2 \hat{e}_{\alpha_1 + \alpha_2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right)$ .

Finally composing on the right with  $\hat{\theta}^{-1}[\sigma_1 \hat{e}_{\alpha_1}]$  leaves  $\text{Ad } \mathbf{E}([\langle \alpha_1, \alpha_2 \rangle]_q \sigma_1 \sigma_2 \hat{e}_{w_1 \alpha_1 + w_2 \alpha_2})$ .  $\square$

Let  $\mathfrak{D} = \{(\mathfrak{d}_i, \hat{\theta}_i)\}$  be a scattering diagram which contains only lines. Fix (possibly trivial) factorizations  $\hat{\theta}_i = \prod_j \hat{\theta}_{ij}$ , where each  $\hat{\theta}_{ij}$  is a finite product of  $\hat{\theta}^\Omega[(-q^{\frac{1}{2}})^n \sigma \hat{e}_{k\alpha}]$ . A *perturbation* of  $\mathfrak{D}$  is a diagram of the form  $\tilde{\mathfrak{D}} = \{(\mathfrak{d}_i + \beta_{ij}, \hat{\theta}_{ij})\}$  for some  $\beta_{ij} \in \mathbb{R}^2$ . Suppose now that  $\mathfrak{D}$  contains only lines through the origin. Then the *asymptotic diagram*  $\mathfrak{D}'_{\text{as}}$  of an arbitrary scattering diagram  $\mathfrak{D}'$  is defined by replacing  $\mathfrak{d} = (\alpha'_0 + \mathbb{R}_{\geq 0} \alpha_0, \hat{\theta}_{\mathfrak{d}}) \in \mathfrak{D}'$  by  $\mathfrak{d}' = (\mathbb{R}_{\geq 0} \alpha_0, \hat{\theta}_{\mathfrak{d}}) \in \mathfrak{D}'_{\text{as}}$ , and  $\mathfrak{d} = (\alpha'_0 + \mathbb{R}_{\geq 0} \alpha_0, \hat{\theta}_{\mathfrak{d}}) \in \mathfrak{D}'$  with  $\mathfrak{d}' = (\mathbb{R} \alpha_0, \hat{\theta}_{\mathfrak{d}}) \in \mathfrak{D}'_{\text{as}}$ . If  $\mathfrak{D}$  is a scattering diagram which contains only lines through the origin, and  $\tilde{\mathfrak{D}}$  some perturbation, then  $S(\mathfrak{D})$  is equivalent to  $S(\tilde{\mathfrak{D}})_{\text{as}}$  (since  $\hat{\theta}_{\pi, S(\tilde{\mathfrak{D}})_{\text{as}}} = \text{Id}$  for a sufficiently large simple loop around the origin).

Let  $R = \mathbb{C}[[t_1, \dots, t_n]]$ . Fix positive, primitive  $\alpha_i \in \Gamma$  (not necessarily distinct) for  $i = 1, \dots, n$ . We will write  $\alpha$  for the vector  $(\alpha_1, \dots, \alpha_n)$ . A *standard* scattering diagram is one of the form

$$\mathfrak{D} = \{(\mathbb{R}\alpha_i, \prod_{w \geq 1} \hat{\theta}[b_{iw} t_i \hat{e}_{w\alpha_i}]), i = 1, \dots, n\}$$

where the coefficients  $b_{iw} \in \mathbb{C}$  vanish for all but finitely many  $w$ . By the above discussion computing  $[\hat{\theta}^{\ell_1}[t_1 \hat{e}_{\alpha_1}], \hat{\theta}^{\ell_2}[t_2 \hat{e}_{\alpha_2}]]$  can be regarded as a special case of the problem of

saturating a standard scattering diagram (after a perturbation). Passing to  $\tilde{R}_k$  we get a truncated diagram  $\mathfrak{D}_k$ , and we can write

$$\prod_{w \geq 1} \hat{\theta}[b_{iw} t_i \hat{e}_{w\alpha_i}] \equiv \text{Ad exp} \left( \sum_{j=1}^k \sum_{w \geq 1} a'_{ijw} \hat{e}_{w\alpha_i} t_i^j \right)$$

for coefficients  $a'_{ijw} \in \mathbb{C}(q^{\pm \frac{1}{2}})$  which vanish for all but finitely many  $w$ . Following the notation of (3.6) there are standard factorizations over  $\tilde{R}_k$  given by

$$\begin{aligned} \text{Ad exp} \left( \sum_{j=1}^k \sum_{w \geq 1} a'_{ijw} \hat{e}_{w\alpha_i} t_i^j \right) &= \prod_J \prod_w \text{Ad exp} \left( (\#J)! a'_{i(\#J)w} \sum_{s \in J} u_{is} \hat{e}_{w\alpha_i} \right) \\ &= \prod_J \prod_w \hat{\theta}[(\#J)! a_{i(\#J)w} \sum_{s \in J} u_{is} \hat{e}_{w\alpha_i}]. \end{aligned}$$

Use these to get a perturbation  $\tilde{\mathfrak{D}}_k = \{(\mathbb{R}\alpha_i + \beta_{iJw}, \hat{\theta}[(\#J)! a_{i(\#J)w} \sum_{s \in J} u_{is} \hat{e}_{w\alpha_i}])\}$ . We construct an increasing sequence of diagrams  $\tilde{\mathfrak{D}}_k^i$ , starting from  $\tilde{\mathfrak{D}}_k^0 = \tilde{\mathfrak{D}}_k$ , which stabilize to  $S(\tilde{\mathfrak{D}}_k)$  for  $i \gg 1$ . We assume inductively that every element of  $\tilde{\mathfrak{D}}_k^i$  is of the form  $(\mathfrak{d}, \hat{\theta}[c_{\mathfrak{d}} u_{I(\mathfrak{d})} \hat{e}_{\alpha_{\mathfrak{d}}}] )$ , where  $c_{\mathfrak{d}} \in \mathbb{C}$ ,  $I(\mathfrak{d}) \subset \{1, \dots, n\} \times \{1, \dots, k\}$  and we set  $u_{I(\mathfrak{d})} = \prod_{(i,j) \in I(\mathfrak{d})} u_{ij}$ . We define a *scattering pair*  $\{\mathfrak{d}_1, \mathfrak{d}_2\} \subset \tilde{\mathfrak{D}}_k^i$  to be a pair of lines or rays such that  $\mathfrak{d}_1, \mathfrak{d}_2 \notin \tilde{\mathfrak{D}}_k^{i-1}$ ,  $\mathfrak{d}_1 \cap \mathfrak{d}_2$  is a single point  $\alpha'_0$  and  $I(\mathfrak{d}_1) \cap I(\mathfrak{d}_2) = \emptyset$ . We let  $\tilde{\mathfrak{D}}_k^{i+1}$  be the union of  $\tilde{\mathfrak{D}}_k^i$  with all the rays of the form

$$(\alpha'_0 + \mathbb{R}_{\geq 0}(\alpha_{\mathfrak{d}_1} + \alpha_{\mathfrak{d}_2}), \hat{\theta}([\langle \alpha_{\mathfrak{d}_1}, \alpha_{\mathfrak{d}_2} \rangle]_q c_{\mathfrak{d}_1} c_{\mathfrak{d}_2} u_{I(\mathfrak{d}_1)} u_{I(\mathfrak{d}_2)} \hat{e}_{\alpha_{\mathfrak{d}_1} + \alpha_{\mathfrak{d}_2}})), \quad (4.1)$$

where  $\{\mathfrak{d}_1, \mathfrak{d}_2\}$  are as above, and we assume (without loss of generality) that the slope of  $\mathfrak{d}_1$  is smaller than the slope of  $\mathfrak{d}_2$ . For a suitably general initial perturbation  $\tilde{\mathfrak{D}}_k$ , we can assume that for all  $i = 1, \dots, nk$  and scattering pairs  $\{\mathfrak{d}_1, \mathfrak{d}_2\}$  there is no further  $\mathfrak{d} \in \tilde{\mathfrak{D}}_k^i$  such that  $\mathfrak{d}_1 \cap \mathfrak{d}_2 \cap \mathfrak{d} \neq \emptyset$ , and  $I(\mathfrak{d}_1) \cap I(\mathfrak{d}_2) \cap I(\mathfrak{d}) = \emptyset$ . We will always make this genericity assumption on  $\tilde{\mathfrak{D}}_k$  in what follows.

**Lemma 4.4.** *The  $\tilde{\mathfrak{D}}_k^i$  stabilize to a scattering diagram  $\tilde{\mathfrak{D}}_k^\infty$  for  $i > nk$ , and  $\tilde{\mathfrak{D}}_k^\infty$  is equivalent to  $S(\tilde{\mathfrak{D}}_k)$ .*

*Proof.* If  $\mathfrak{d}_1, \mathfrak{d}_2 \in \tilde{\mathfrak{D}}_k^i \setminus \tilde{\mathfrak{D}}_k^{i-1}$  for  $i > nk$ , then  $I(\mathfrak{d}_1) \cap I(\mathfrak{d}_2) \neq \emptyset$ , so  $u_{I(\mathfrak{d}_1)} u_{I(\mathfrak{d}_2)} = 0$ , and  $\tilde{\mathfrak{D}}_k^i = \tilde{\mathfrak{D}}_k^{i+1}$ . We set  $\tilde{\mathfrak{D}}_k^\infty = \tilde{\mathfrak{D}}_k^i$  for  $i > nk$ . If  $\tilde{\mathfrak{D}}_k^\infty \neq S(\tilde{\mathfrak{D}}_k)$ , we could find a simple loop  $\pi$  around a point  $\alpha'_0 \in \text{Sing}(\tilde{\mathfrak{D}}_k^\infty)$  for which  $\hat{\theta}_{\pi, \tilde{\mathfrak{D}}_k^\infty} \neq \text{Id}$ . By construction this implies that there are two rays  $\mathfrak{d}_1, \mathfrak{d}_2 \in \tilde{\mathfrak{D}}_k^\infty$  for which the ray (4.1) does not belong to  $\tilde{\mathfrak{D}}_k^\infty$ , a contradiction.  $\square$

Let  $(\mathfrak{d}, \hat{\theta}_{\mathfrak{d}})$  be an element of some  $\tilde{\mathfrak{D}}_k^i$ . We associate to  $\mathfrak{d}$  an immersed graph  $\Upsilon_{\mathfrak{d}} \subset \mathbb{R}^2$ , with both bounded and unbounded edges, which is either trivalent or a single line.  $\Upsilon_{\mathfrak{d}}$  is constructed inductively; if  $\mathfrak{d}$  is a line then  $\Upsilon_{\mathfrak{d}} = \mathfrak{d}$ . Otherwise by construction  $\mathfrak{d}$  arises uniquely from the scattering of a pair  $\{\mathfrak{d}_1, \mathfrak{d}_2\} \subset \tilde{\mathfrak{D}}_k^{i-1}$ , with  $\mathfrak{d}_1 \cap \mathfrak{d}_2 = \alpha'_0$ . We let

$$\Upsilon_{\mathfrak{d}} = (\Upsilon_{\mathfrak{d}_1} \setminus (\alpha'_0 + \mathbb{R}_{\geq 0} \alpha_{\mathfrak{d}_1})) \cup (\Upsilon_{\mathfrak{d}_2} \setminus (\alpha'_0 + \mathbb{R}_{\geq 0} \alpha_{\mathfrak{d}_2})) \cup (\alpha'_0 + \mathbb{R}_{\geq 0}(\alpha_{\mathfrak{d}_1} + \alpha_{\mathfrak{d}_2})).$$

The induction makes sense since  $\tilde{\mathfrak{D}}_k^0$  contains only lines. Suppose that  $\Upsilon_{\mathfrak{d}}$  is not a line. Then by construction it contains a finite number of unbounded edges, including the *outgoing* edge  $\mathfrak{d}$ . The other (*incoming*) unbounded edges are all contained in the lines  $\mathfrak{d}_{iJw}$ . By standard arguments we can think of  $\Upsilon_{\mathfrak{d}}$  as a rational tropical curve. More

precisely there exists a unique equivalence class of parametrized rational tropical curves  $h : \tilde{\Upsilon}_{\mathfrak{d}} \rightarrow \mathbb{R}^2$  where  $(\tilde{\Upsilon}_{\mathfrak{d}}, w_{\tilde{\Upsilon}_{\mathfrak{d}}})$  is a simply connected trivalent weighted graph (with both bounded and unbounded edges, endowed with its standard topology),  $h$  is a proper map with  $h(\tilde{\Upsilon}_{\mathfrak{d}}) = \Upsilon_{\mathfrak{d}}$ ,  $w_{\tilde{\Upsilon}_{\mathfrak{d}}}(E) = w$  when  $E$  is an unbounded edge mapping to a ray contained in  $\mathfrak{d}_{iJw}$ , and the unbounded direction of  $h(E)$  is  $-\alpha_i$ . We will not be very careful in distinguishing  $h : \tilde{\Upsilon}_{\mathfrak{d}} \rightarrow \mathbb{R}^2$  from its image. By construction, we have

**Lemma 4.5.** *There is a bijective correspondence between elements  $(\mathfrak{d}, \hat{\theta}_{\mathfrak{d}}) \in \tilde{\mathfrak{D}}_k^i$  and rational tropical curves  $\Upsilon_{\mathfrak{d}}$  whose unbounded edges either coincide with  $\mathfrak{d}$  or are contained in  $\mathfrak{d}_{iJw}$  (we prescribe the multiplicity of such an unbounded edge to be  $w$ ).*

Let  $V$  be a vertex of  $\tilde{\Upsilon}_{\mathfrak{d}}$ , and choose two incident edges  $E_1, E_2$ . Let  $v_1, v_2$  denote the primitive vectors in the direction of  $h(E_1), h(E_2)$ . The *tropical multiplicity* of  $h$  at  $V$  is defined as  $\mu(h, V) = w_{\tilde{\Upsilon}_{\mathfrak{d}}}(E_1)w_{\tilde{\Upsilon}_{\mathfrak{d}}}(E_2)|\det(v_1, v_2)|$ . This is well defined by the balancing condition  $\sum_i w_{\tilde{\Upsilon}_{\mathfrak{d}}}(E_i)v_i = 0$  where  $E_i$  are the incident edges at  $V$  and the  $v_i$  primitive vectors in the direction of  $h(E_i)$  ( $i = 1 \dots 3$ ). Then one defines the *tropical multiplicity* of  $h$  as  $\mu(h) = \prod_V \mu(h, V)$ , the product over all trivalent vertices. The *Block-Göttsche multiplicity* at  $V$  is defined as  $\mu_q(h, V) = [\mu(h, V)]_q$ . Similarly one sets  $\mu_q(h) = \prod_V \mu_q(h, V)$ . Using this notion we can reconstruct  $\hat{\theta}_{\mathfrak{d}}$  from  $\Upsilon_{\mathfrak{d}}$  with its tropical structure.

**Lemma 4.6.** *Fix  $(\mathfrak{d}, \hat{\theta}_{\mathfrak{d}}) \in \tilde{\mathfrak{D}}_k^i$ . Let  $\alpha_{\text{out}} = \sum_{i,J,w} w\alpha_i$ , summing over all  $i, J, w$  for which there exists an unbounded incoming edge of  $\Upsilon_{\mathfrak{d}}$  contained in  $\mathfrak{d}_{iJw}$ . Then*

$$\hat{\theta}_{\mathfrak{d}} = \hat{\theta}[\mu_q(\Upsilon_{\mathfrak{d}}) \left( \prod_{i,J,w} (\#J)! a_{i(\#J)w} \prod_{s \in J} u_{is} \right) \hat{e}_{\alpha_{\text{out}}}] .$$

*Proof.* Consider the statement: for two rays  $\mathfrak{d}_1, \mathfrak{d}_2 \in \tilde{\mathfrak{D}}_k^j$ ,  $j < i$ , with the slope of  $\mathfrak{d}_1$  smaller than the slope of  $\mathfrak{d}_2$ , incoming at  $V$ , and scattering in a ray  $\mathfrak{d}'$ , one has  $\mu(\Upsilon_{\mathfrak{d}'}, V) = \langle \alpha_{\mathfrak{d}_1}, \alpha_{\mathfrak{d}_2} \rangle$ . One checks this for  $j = 0$ , and by induction it holds for all  $j < i$  (the point being of course that  $\langle -, - \rangle$  is the same as  $\det(-, -)$ ). The statement about  $\hat{\theta}_{\mathfrak{d}}$  and  $\Upsilon_{\mathfrak{d}}$  then follows by induction on  $i$  and Lemma 4.3.  $\square$

Fix a *weight vector*  $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_n)$ , where each  $\mathbf{w}_i$  is a collection of integers  $w_{ij}$  (for  $1 \leq i \leq n$  and  $1 \leq j \leq l_i$ ) such that  $1 \leq w_{i1} \leq w_{i2} \leq \dots \leq w_{il_i}$ . For  $1 \leq j \leq l_i$  choose a general collection of parallel lines  $\mathbf{c}_{ij}$  in the direction  $\alpha_i$ . We attach the weight  $w_{ij}$  to the line  $\mathbf{c}_{ij}$ , and think of the lines  $\mathbf{c}_{ij}$  as incoming unbounded edges for connected, rational tropical curves  $\Upsilon \subset \mathbb{R}^2$ . We prescribe that such curves  $\Upsilon$  have a single additional outgoing unbounded edge. Let us denote by  $\mathcal{S}(\mathbf{w}, \mathbf{c}_{ij})$  the finite set of such tropical curves  $\Upsilon$  (for a general, fixed choice of ends  $\mathbf{c}_{ij}$ ).

**Definition 4.7.** *We denote by  $\hat{N}_{\alpha}^{\text{trop}}(\mathbf{w}) = \#^{\mu_q} \mathcal{S}(\mathbf{w}, \mathbf{c}_{ij})$  the  $q$ -deformed tropical count of curves  $\Upsilon$  as above, i.e. the number of elements of  $\mathcal{S}(\mathbf{w}, \mathbf{c}_{ij})$  counted with the Block-Göttsche multiplicity  $\mu_q$ .*

By the results of [IM]  $\hat{N}_{\alpha}^{\text{trop}}(\mathbf{w})$  is independent of a general choice of  $\mathbf{c}_{ij}$ , so in particular it makes sense to drop  $\mathbf{c}_{ij}$  from the notation.

**Corollary 4.8.** *Let  $(\mathfrak{d}, \hat{\theta}_{\mathfrak{d}}) \in \mathcal{S}(\mathfrak{D})$ . Then*

$$\hat{\theta}_{\mathfrak{d}} = \text{Ad exp} \left( \sum_{\mathbf{w}} \sum_{\mathbf{k}} \frac{\hat{N}_{\alpha}^{\text{trop}}(\mathbf{w})}{|\text{Aut}(\mathbf{w}, \mathbf{k})|} \left( \prod a_{ik_{ij}w_{ij}} t_i^{k_{ij}} \right) \frac{\hat{e}_{\sum_i |\mathbf{w}_i| \alpha_i}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right),$$

where the first sum is over weight vectors  $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_n)$  for which  $\sum_i |\mathbf{w}_i|_{\alpha_i} \in \mathfrak{d}$ , the second sum is over collections  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_n)$  of vectors  $\mathbf{k}_i$  with the same length as  $\mathbf{w}_i$ , such that  $k_{ij} \leq k_{i(j+1)}$  if  $w_{ij} = w_{i(j+1)}$ , and  $\text{Aut}(\mathbf{w}, \mathbf{k})$  is the product of all stabilizers  $\text{Aut}(\mathbf{w}_i, \mathbf{k}_i)$  in the symmetric group.

*Proof.* It is enough to prove the statement modulo  $(t_1^{k+1}, \dots, t_n^{k+1})$  for all  $k$ . By the above discussion we know that  $\hat{\theta}_{\mathfrak{d}}$  equals the product of operators  $\hat{\theta}_{\mathfrak{d}'}$  for  $\mathfrak{d}' \in \tilde{\mathfrak{D}}_k^\infty$  parallel to  $\mathfrak{d}$ . By Lemma 4.5 each  $\mathfrak{d}'$  corresponds to a unique set  $\{\mathfrak{d}_{iJw}\} \subset \tilde{\mathfrak{D}}_k^0$ . The indices  $w$  occurring in  $\{\mathfrak{d}_{iJw}\}$  for  $i = 1, \dots, n$  then define the collection of weight vectors  $\mathbf{w}$ . The indices  $\{J\}$  on the other hand define a collection of subsets of  $\{1, \dots, n\} \times \{1, \dots, k\}$ ; their cardinalities determine the collection of vectors  $\mathbf{k}$ . By Lemma 4.6 the product of all  $\hat{\theta}_{\mathfrak{d}'}$  for fixed  $\{\mathfrak{d}_{iJw}\}$  equals  $\text{Ad exp} \left( \#^{\mu_q} \mathcal{S}(\mathbf{w}, \{\mathfrak{d}_{iJw}\}) \left( \prod k_{ij}! a_{ik_{ij}w_{ij}} \prod_J u_J \right) (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1} \hat{e}_{\sum_i |\mathbf{w}_i|_{\alpha_i}} \right)$ . The key point is that by [IM]  $\#^{\mu_q} \mathcal{S}(\mathbf{w}, \{\mathfrak{d}_{iJw}\})$  only depends on  $\{\mathfrak{d}_{iJw}\}$  through  $\mathbf{w}$  and equals  $\hat{N}_{\alpha}^{\text{trop}}(\mathbf{w})$ . From here the proof proceeds as in [GPS] Theorem 2.8.  $\square$

## 5. SOME $q$ -DEFORMED GROMOV-WITTEN INVARIANTS

We follow the notation of the previous section. Consider the scattering diagram over  $\mathbb{C}[[s, t]]$  given by  $\mathfrak{D}' = \{(\mathbb{R}\alpha_1, \hat{\theta}^{\ell_1}[s\hat{e}_{\alpha_1}]), (\mathbb{R}\alpha_2, \hat{\theta}^{\ell_2}[t\hat{e}_{\alpha_2}])\}$  for  $\ell_i \in \mathbb{N}$ . The saturation  $S(\mathfrak{D}')$  can be recovered from  $S(\mathfrak{D})$  for  $\mathfrak{D}$  the diagram over  $\mathbb{C}[[s_1, \dots, s_{\ell_1}, t_1, \dots, t_{\ell_2}]]$  given by

$$\mathfrak{D} = \{(\mathbb{R}\alpha_1, \hat{\theta}[s_1 e_{\alpha_1}]), \dots, (\mathbb{R}\alpha_1, \hat{\theta}[s_{\ell_1} e_{\alpha_1}]), (\mathbb{R}\alpha_2, \hat{\theta}[t_1 \hat{e}_{\alpha_2}]), \dots, (\mathbb{R}\alpha_2, \hat{\theta}[t_{\ell_2} \hat{e}_{\alpha_2}])\}.$$

Recall

$$\hat{\theta}[s_i \hat{e}_{\alpha_1}] = \text{Ad exp} \left( - \sum_{j=1}^k \frac{s_i^j \hat{e}_{j\alpha_1}}{j((-q^{\frac{1}{2}})^j - (-q^{-\frac{1}{2}})^{-j})} \right),$$

so in the notation of section 4 we have

$$a'_{ijj} = \frac{(-1)^j}{j((-q^{\frac{1}{2}})^j - (-q^{-\frac{1}{2}})^{-j})}, \quad a_{ijj} = ((-q^{\frac{1}{2}}) - (-q^{-\frac{1}{2}})) a'_{ijj} = \frac{(-1)^{j-1}}{j[j]_q}$$

while all other  $a_{ijw}$  vanish. The same computation holds for  $\hat{\theta}[t_i \hat{e}_{\alpha_2}]$ . For the diagram  $\mathfrak{D}$  the  $(\ell_1 + \ell_2)$ -tuple of weight vectors  $\mathbf{w}$  required by Corollary 4.8 can be parametrized in a more convenient way. We first fix a pair of ordered partitions  $(\mathbf{P}_1, \mathbf{P}_2)$  of length  $\ell_1, \ell_2$  respectively. The part  $\mathbf{P}_{1i}$  determines the size of a weight vector corresponding to  $(\mathbb{R}\alpha_1, \hat{\theta}[s_i e_{\alpha_1}])$ , and similarly for  $\mathbf{P}_{2i}$ . So we can enumerate instead in terms of just a pair of weight vectors  $(\mathbf{w}_1, \mathbf{w}_2)$ , with  $\mathbf{w}_i = \{w_{i1} \leq \dots \leq w_{i\ell_i}\}$ , plus a pair of *compatible set partitions*  $(I_{1,\bullet}, I_{2,\bullet})$  of the index sets  $\{1, \dots, \ell_1\}, \{1, \dots, \ell_2\}$ , that is set partitions  $I_{i,\bullet}$  for which  $\sum_{s \in I_{ij}} w_{is} = \mathbf{P}_{ij}$ . Let us denote by  $\#\{I_{i,\bullet}, \mathbf{P}_i | \mathbf{w}_i\}$  the number of set partitions of  $\mathbf{w}_i$  which are compatible with  $\mathbf{P}_i$ , and introduce the  $q$ -deformed ramification factors

$$\hat{R}_{\mathbf{P}_i | \mathbf{w}_i, q} = \prod_j \frac{(-1)^{w_{ij}-1}}{w_{ij}[w_{ij}]_q} \#\{I_{i,\bullet}, \mathbf{P}_i | \mathbf{w}_i\}.$$

Then we may apply Corollary 4.8 to see that the operators appearing in the saturation of  $\mathfrak{D}$  are given by

$$\hat{\theta}_{a_1\alpha_1 + a_2\alpha_2} = \text{Ad exp} \left( \sum_{|\mathbf{P}_1|=ka_1} \sum_{|\mathbf{P}_2|=ka_2} \sum_{\mathbf{w}} \prod_{i=1}^2 \frac{\hat{R}_{\mathbf{P}_i | \mathbf{w}_i}}{|\text{Aut}(\mathbf{w}_i)|} \hat{N}_{(\alpha_1, \alpha_2)}^{\text{trop}}(\mathbf{w}) s^{\mathbf{P}_1} t^{\mathbf{P}_2} \frac{\hat{e}_{k(a_1\alpha_1 + a_2\alpha_2)}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right). \quad (5.1)$$

By specialization we finally obtain the analogue of the basic GPS identity (2.4).

**Proposition 5.1.** *The operators in the slope ordered expansion for  $[\hat{\theta}^{\ell_1}[t\hat{e}_{\alpha_1}], \hat{\theta}^{\ell_2}[t\hat{e}_{\alpha_2}]]$  are obtained by setting  $s_i = t_j = t$  in (5.1) for  $i, j = 1, \dots, n$ .*

By comparing (5.1) with the formulae (2.3)-(2.6) it is natural to introduce a class of putative  $q$ -deformed Gromov-Witten invariants.

**Definition 5.2.** *We define a  $q$ -deformation of the invariant  $N[(\mathbf{P}_1, \mathbf{P}_2)]$  by*

$$\hat{N}[(\mathbf{P}_1, \mathbf{P}_2)] = \sum_{\mathbf{w}} \prod_{i=1}^2 \frac{\hat{R}_{\mathbf{P}_i|\mathbf{w}_i}}{|\text{Aut}(\mathbf{w}_i)|} \hat{N}_{(\alpha_1, \alpha_2)}^{\text{trop}}(\mathbf{w}).$$

In the special case when the vector  $(\mathbf{P}_1, \mathbf{P}_2)$  is *primitive* there is a more straightforward candidate for  $\hat{N}[(\mathbf{P}_1, \mathbf{P}_2)]$ . Consider the complete bipartite quiver  $\mathcal{K}(\ell_1, \ell_2)$  endowed with its natural notion of stability ([RSW] section 4). The pair  $(\mathbf{P}_1, \mathbf{P}_2)$  induces a dimension vector for  $\mathcal{K}(\ell_1, \ell_2)$ , and we denote by  $\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2)$  the resulting moduli space of stable representations. By a result of Reineke and Weist ([RW] Corollary 9.1) we have

$$N[(\mathbf{P}_1, \mathbf{P}_2)] = \chi(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2)).$$

Accordingly a natural choice would be to define a  $q$ -deformed Gromov-Witten invariant as the Poincaré polynomial  $P(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))(q) = \sum_j b_j(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))q^{\frac{j}{2}}$  (recall that in fact the odd Betti numbers  $b_{2j+1}$  vanish). However it makes more sense to have a notion which is invariant under the change of variable  $q \mapsto q^{-1}$ . Thus we set

$$\hat{N}'[(\mathbf{P}_1, \mathbf{P}_2)] = \hat{P}(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))(q) := q^{-\frac{1}{2} \dim \mathcal{M}(\mathbf{P}_1, \mathbf{P}_2)} P(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))(q).$$

In the notation of [MR]  $\hat{P}(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))(q) = P([\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2)]_{\text{vir}})(q)$ , where for  $X$  an irreducible smooth algebraic variety one sets  $[X]_{\text{vir}} = q^{-\frac{1}{2} \dim X} [X]$ , a Laurent polynomial with coefficients in the Grothendieck ring of varieties.

**Theorem 5.3.** *The two choices of quantization coincide:  $\hat{N}'[(\mathbf{P}_1, \mathbf{P}_2)] = \hat{N}[(\mathbf{P}_1, \mathbf{P}_2)]$ .*

*Proof.* We will reduce the statement to a representation-theoretic formula due to Man-schot, Pioline and Sen [MPS]. A *refinement* of  $(\mathbf{P}_1, \mathbf{P}_2)$  is a pair of sets of integers  $(k^1, k^2) = (\{k_{w,i}^1\}, \{k_{w,j}^2\})$  such that for  $i = 1, \dots, \ell_1$  and  $j = 1, \dots, \ell_2$  we have  $p_{1i} = \sum_w w k_{w,i}^1$ ,  $p_{2j} = \sum_w w k_{w,j}^2$ . We denote refinements by  $(k^1, k^2) \vdash (\mathbf{P}_1, \mathbf{P}_2)$ . A fixed refinement  $k^i$  induces a weight vector  $\mathbf{w}(k^i) = (w_{i1}, \dots, w_{it_i})$  of length  $t_i = \sum_w m_w(k^i)$ , by  $w_{ij} = w$  for all  $j = \sum_{r=1}^{w-1} m_r(k^i) + 1, \dots, \sum_{r=1}^w m_r(k^i)$ . By a combinatorial argument contained in the proof of [RSW] Lemma 4.2 we may rearrange (5.2) as

$$\hat{N}[(\mathbf{P}_1, \mathbf{P}_2)] = \sum_{(k^1, k^2) \vdash (\mathbf{P}_1, \mathbf{P}_2)} \prod_{i=1}^2 \prod_{j=1}^{\ell_i} \prod_w \frac{(-1)^{k_{w,j}^i(w-1)}}{k_{w,j}^i! w^{k_{w,j}^i} [w]_q^{k_{w,j}^i}} \hat{N}_{(\alpha_1, \alpha_2)}^{\text{trop}}(\mathbf{w}(k^1), \mathbf{w}(k^2)).$$

From now we follow closely the treatment in [RSW] section 4; in particular we will formulate the MPS result using the infinite bipartite quiver  $\mathcal{N}$  introduced there, with vertices  $\mathcal{N}_0 = \{i_{(w,m)} \mid (w,m) \in \mathbb{N}^2\} \cup \{j_{(w,m)} \mid (w,m) \in \mathbb{N}^2\}$  and arrows  $\mathcal{N}_1 = \{\alpha_1, \dots, \alpha_{w \cdot w'} : i_{(w,m)} \rightarrow j_{(w',m')}, \forall w, w', m, m' \in \mathbb{N}\}$ . The quiver  $\mathcal{N}$  comes with a notion of stability in terms of a slope function  $\mu$  (there is no possible confusion with the tropical multiplicity  $\mu$  since the latter will not appear in this proof). Recall that  $(k^1, k^2)$  induces a *thin* (i.e. type one) dimension vector  $d(k^1, k^2)$  for  $\mathcal{N}$ , so we get a moduli space of stable *abelian* representations  $\mathcal{M}_{d(k^1, k^2)}(\mathcal{N})$ . Following the argument leading to equation (1) of [RSW],

and after rearranging to pass from  $P$  to  $\hat{P}$ , the MPS formula for Poincaré polynomials in this setup can be expressed as

$$\hat{P}(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))(q) = \sum_{(k^1, k^2) \vdash (\mathbf{P}_1, \mathbf{P}_2)} \prod_{i=1}^2 \prod_{j=1}^{\ell_i} \prod_w \frac{(-1)^{k_{w,j}^i(w-1)}}{k_{w,j}^i! w^{k_{w,j}^i} [w]_q^{k_{w,j}^i}} \hat{P}(\mathcal{M}_{d(k^1, k^2)}(\mathcal{N}))(q). \quad (5.2)$$

Indeed in the general case (for  $(\mathbf{P}_1, \mathbf{P}_2)$  not necessarily primitive) one can rewrite the MPS formula as

$$\frac{[R_{(\mathbf{P}_1, \mathbf{P}_2)}^{\text{sst}}(K(\ell_1, \ell_2))]_{\text{vir}}}{[\text{GL}(\mathbf{P}_1, \mathbf{P}_2)]_{\text{vir}}} = \sum_{(k^1, k^2) \vdash (\mathbf{P}_1, \mathbf{P}_2)} \prod_{i=1}^2 \prod_{j=1}^{\ell_i} \prod_w \frac{(-1)^{k_{w,j}^i(w-1)}}{k_{w,j}^i! w^{k_{w,j}^i} [w]_q^{k_{w,j}^i}} \frac{[R_{d(k^1, k^2)}^{\text{sst}}(\mathcal{N})]_{\text{vir}}}{[(\mathbb{C}^*)^{|k^1|+|k^2|}]_{\text{vir}}}$$

where we have denoted by  $R^{\text{sst}}(-)$  the semistable loci, and by  $\text{GL}(\mathbf{P}_1, \mathbf{P}_2)$  the usual basechange group corresponding to a dimension vector; this is explained in [MR] section 8.1. The claim of the Lemma then follows from the identity

$$\hat{P}(\mathcal{M}_{d(k^1, k^2)}(\mathcal{N}))(q) = \hat{N}_{(\alpha_1, \alpha_2)}^{\text{trop}}(\mathbf{w}(k^1), \mathbf{w}(k^2)). \quad (5.3)$$

To prove this let  $\mathcal{Q} \subset \mathcal{N}$  denote the finite subquiver which is the support of  $(k^1, k^2)$ . Consider the lattice  $\tilde{\Gamma} = \mathbb{Z}\mathcal{Q}_0$  endowed with the bilinear form  $\langle -, - \rangle$  which is the antisymmetrization of the Euler form of  $\mathcal{Q}_0$ . We will write  $\tilde{\Gamma}_\mu^+$  for the subsemigroup of dimension vectors of slope  $\mu$ . Let  $R = \mathbb{C}[[t_{i_{(w', m')}}, t_{j_{(w, m)}}]]$ . We work in the group  $\mathbb{U}_{\tilde{\Gamma}, R}$  and consider the product of operators

$$\prod_{j_{(w, m)} \in \mathcal{Q}_0} \hat{\theta}[t_{j_{(w, m)}} \hat{e}_{j_{(w, m)}}] \prod_{i_{(w', m')} \in \mathcal{Q}_0} \hat{\theta}[t_{i_{(w', m')}} \hat{e}_{i_{(w', m')}}]. \quad (5.4)$$

By [R] Lemma 4.3, (5.4) can be expressed as an ordered product  $\prod_{\mu \in \mathbb{Q}}^{\leftarrow} \text{Ad } \tilde{P}_\mu$  where  $\tilde{P}_\mu \in \hat{\mathfrak{g}}_q$  is an element of the form  $\sum_{d \in \tilde{\Gamma}_\mu^+} \tilde{p}_d(q) t^d \hat{e}_d$  for some  $\tilde{p}_d(q) \in \mathbb{Q}(q)$ . The  $\tilde{P}_\mu$  are characterized in terms of the Harder-Narashtiman recursion. By the definition of the  $\tilde{P}_\mu$ , using that  $(\mathbf{P}_1, \mathbf{P}_2)$  is primitive and  $d(k^1, k^2)$  is thin, one shows that the first nontrivial term in  $\tilde{P}_{\mu(d(k^1, k^2))}$  is  $\tilde{p}_{d(k^1, k^2)}(q) t^{d(k^1, k^2)} \hat{e}_{d(k^1, k^2)}$  (as a term of smaller degree would imply the existence of a subrepresentation of the  $d(k^1, k^2)$ -dimensional representation having the same slope). By the remark following [R] Proposition 4.5 we have in fact<sup>3</sup>

$$\tilde{p}_{d(k^1, k^2)}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1} \hat{P}(\mathcal{M}_{d(k^1, k^2)}(\mathcal{N}))(q).$$

On the other hand by an argument contained in the proof of [RSW] Proposition 4.3 we can find a change of variables which preserves slopes and reduces the calculation of the  $\tilde{P}_\mu$  for (5.4) to the problem of saturating a scattering diagram for  $\mathbb{U}_{\Gamma, R}$  with  $\Gamma = \mathbb{Z}^2$  (endowed with its standard antisymmetric bilinear form). We can then combine the rest of the proof of [RSW] Proposition 4.3 with Lemma 4.6 above to compute the first nontrivial term in  $\tilde{P}_{\mu(k^1, k^2)}$  as  $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1} \hat{N}_{(\alpha_1, \alpha_2)}^{\text{trop}}(\mathbf{w}(k^1), \mathbf{w}(k^2))$ . Notice that there are no ramification factors precisely because  $d(k^1, k^2)$  is thin and coprime, and we are computing the first nontrivial term. Matching the two answers gives (5.3).  $\square$

<sup>3</sup>To compare with the results of [R] section 4 one must take into account the different convention for the  $q$ -deformed product. Our functions  $\tilde{P}_\mu, \tilde{p}_d(q)$  are precisely what happens when one replaces the product of [R] Definition 3.1 with our (3.9) (beware that in *ibid.* the notation  $\langle -, - \rangle$  denotes the Euler form, *not* its antisymmetrization).

**Remark.** Theorem 4.1 in [RSW] shows that the MPS formula (going from nonabelian to abelian representations) for  $\chi(\mathcal{M}(\mathbf{P}_1, \mathbf{P}_2))$  and coprime  $(\mathbf{P}_1, \mathbf{P}_2)$  is dual to a standard degeneration formula in Gromov-Witten theory (going from incidence conditions to tangency conditions). The proof of Lemma 5.3 gives another interpretation of the MPS formula at the level of Poincaré polynomials, as a compatibility condition between two natural  $q$ -deformations of the invariant  $N[(\mathbf{P}_1, \mathbf{P}_2)]$ .

**Remark.** When  $(\mathbf{P}_1, \mathbf{P}_2)$  is type one ( $\mathbf{P}_{ij} = 1$ ), by Definition 5.2 we have  $\widehat{N}[(\mathbf{P}_1, \mathbf{P}_2)] = \widehat{N}_{(\alpha_1, \alpha_2)}^{\text{trop}}(\mathbf{P}_1, \mathbf{P}_2)$ . Conjecture 89 in [SG] predicts that some invariants  $\overline{N}_{\text{trop}}^{L, \delta}$  of the same form of the  $\widehat{N}_{(\alpha_1, \alpha_2)}^{\text{trop}}(\mathbf{P}_1, \mathbf{P}_2)$  can be expressed (through a BPS transformation) in terms of the  $\chi_{-y}$  genus of relative Hilbert scheme of points of a linear system of curves on a smooth polarized toric surface  $(S, L)$  (with  $L$  sufficiently ample with respect to  $\delta$ ). One could ask if a similar statement may hold for  $\widehat{N}[(\mathbf{P}_1, \mathbf{P}_2)]$ . We hope to return to this question in future work.

## REFERENCES

- [BG] F. Block and L. Göttsche, *Refined Severi degrees*, to appear. See also <http://www.newton.ac.uk/programmes/MOS/seminars/062716301.html>
- [GM] A. Gathmann and H. Markwig, *The numbers of tropical plane curves through points in general position*, Journal für die reine und angewandte Mathematik 602 (2007), 155-177.
- [SG] L. Göttsche and V. Shende, *Refined curve counting on complex surfaces*, arXiv:1208.1973.
- [GPS] M. Gross, R. Pandharipande and B. Siebert, *The tropical vertex*, Duke Math. J. **153**, no. 2, 297-362 (2010).
- [IM] I. Itenberg and G. Mikhalkin, *On Block-Göttsche multiplicities for planar tropical curves*, arXiv:1201.0451.
- [KS] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435. See also M. Kontsevich and Y. Soibelman, *Motivic Donaldson-Thomas invariants: summary of results*, Mirror symmetry and tropical geometry, 55-89, Contemp. Math., 527, Amer. Math. Soc., Providence, RI, 2010.
- [MPS] J. Manschot, B. Pioline and A. Sen, *Wall-crossing from Boltzmann Black hole halos*, J. High Energy Phys. **1107**, 059 (2011).
- [M] G. Mikhalkin, *Enumerative tropical algebraic geometry in  $\mathbb{R}^2$* . J. Amer. Math. Soc. **18**, 313-377 (2005).
- [MR] S. Mozgovoy and M. Reineke, *Abelian quiver invariants and marginal wall-crossing*, arXiv:1212.0410.
- [P] B. Pioline, *Four ways across the wall*, proceedings of the workshop “Algebra, Geometry and Mathematical Physics”, Tjärnö, Sweden, October 2010. arXiv:1103.0261v2 [hep-th].
- [R] M. Reineke, *Poisson automorphisms and quiver moduli*, J. Inst. Math. Jussieu **9**, 653-667 (2010).
- [RSW] M. Reineke, J. Stoppa and T. Weist, *MPS formula for quiver moduli and refined GW/Kronecker correspondence*, Geometry & Topology **16**, no. 4, 2097-2134 (2012).
- [RW] M. Reineke and T. Weist, *Refined GW/Kronecker correspondence*, Math. Ann. DOI 10.1007/s00208-012-0778-0. arXiv:1103.5283.

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